Compound Stresses and Strains

→ 2·D Systems

→ Stress at a point on a plane

→ Principal stresses and principal planes

→ Mohr's circle of stress and their applications

→ Two-D stress-strain system

→ Principal strains and principal axis of strain

Principal stresses

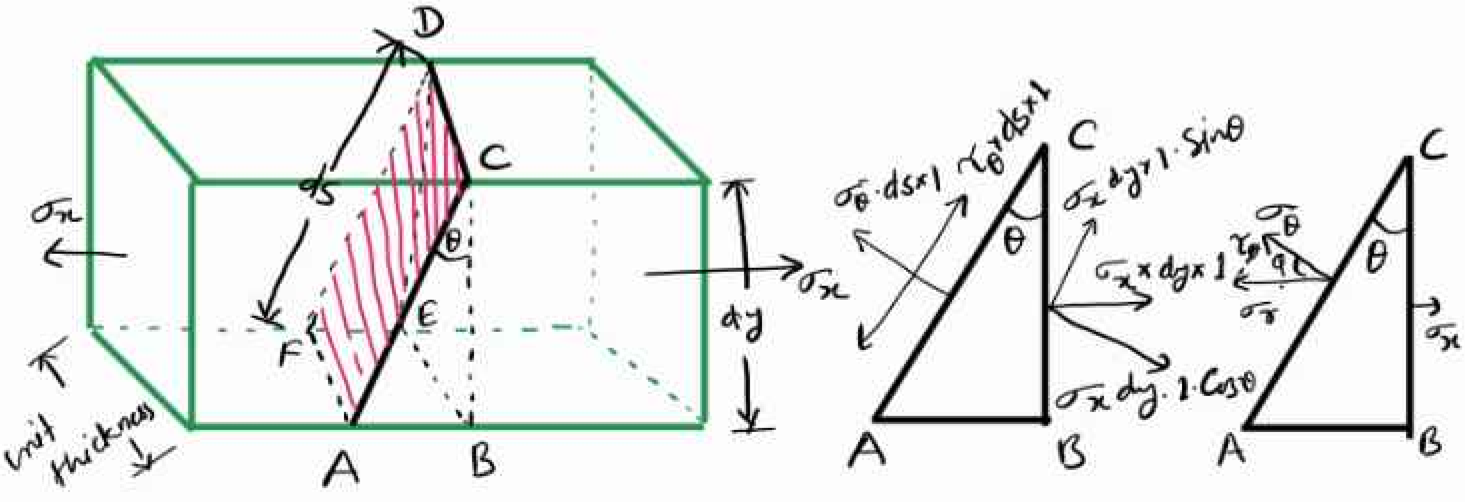
The normal stresses, acting on a principal plane, are known as principal stresses.

Principal planes

The planes, which have no shear stress, are known as principal planes.

Direct stress condition → If a member subjected to a direct stress in one plane

A bar be experienced by an external loading P which results a direct tensile stress σ_x along its lengths.



→ On transverse section BCDE will have a pure normal stress acting on it.

→ A stress acting on an arbitrary plane ACDF inclined at an angle θ with the vertical plane will have two components.

If the bar is imagined cut through the section ACDF, each portion of the bar is also in equilibrium under the action of forces due to the stresses devloped. For convenience, a triangular prismatic element ABCDEF containing the plane ACDF can be taken for the force analysis

In the figure, it is given that

dy = length of the side BC

ds = length of the side AC

σ_x = Normal stress acting on the plane BCDE

σ_θ = Normal stress acting on the plane ACDF

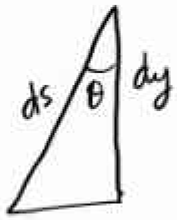
τ_θ = tangential (or) shear stress acting on the plane ACDF.

Assume a unit thickness of the prism and equate the forces along normal and tangential directions to the plane ACDF of the prism for its equilibrium.

$$\sigma_\theta \cdot ds - \sigma_x \, dy . \cos\theta = 0$$

$$\sigma_\theta = \frac{\sigma_x \, dy . \cos\theta}{ds} = \frac{\sigma_x \, dy \cos\theta}{\frac{dy}{\cos\theta}}$$

$$\boxed{\sigma_\theta = \sigma_x \cos^2\theta}$$



$$\cos\theta = \frac{dy}{ds}$$

$$\frac{dy}{\cos\theta} = ds$$

$$\tau_\theta \cdot ds + \sigma_x \, dy \sin\theta = 0$$ { τ_θ is clockwise direction }

$$\tau_\theta = -\frac{\sigma_x \, dy \sin\theta}{ds} = -\frac{\sigma_x \, dy \sin\theta}{dy/\cos\theta} = -\sigma_x \sin\theta \cdot \cos\theta$$

$$\tau_\theta = -\frac{1}{2}\sigma_x \sin 2\theta$$ {Clockwise direction}

$$\boxed{\tau_\theta = \frac{1}{2}\sigma_x \sin 2\theta}$$ {Counterclockwise direction}

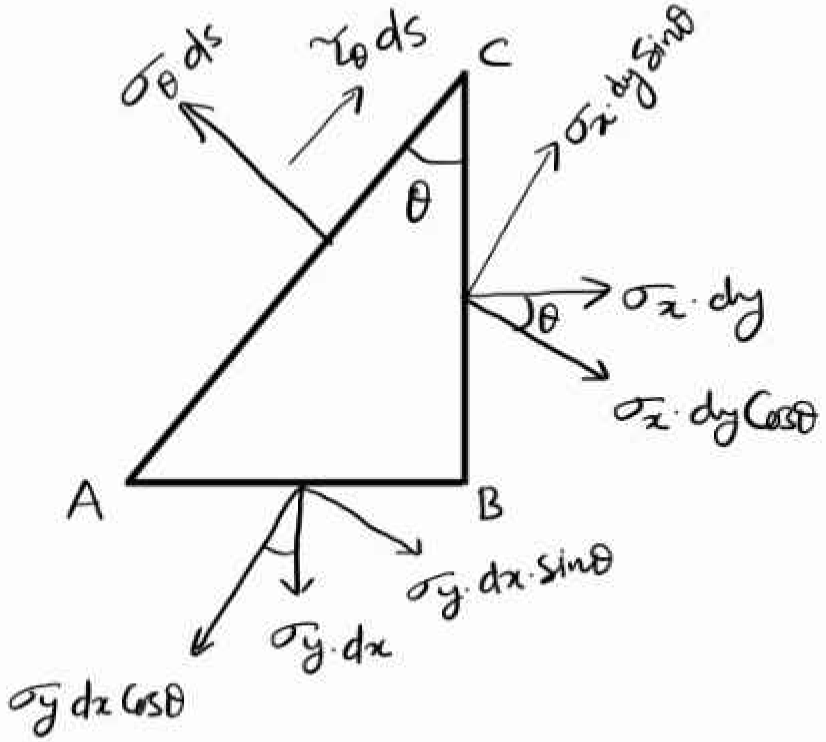
When $\theta = 0^\circ$, $\sigma_\theta = \sigma_x$ and $\tau_\theta = 0$

When $\theta = 45^\circ$, $\sigma_\theta = \frac{\sigma_x}{2}$ and $\tau_\theta = -\frac{\sigma_x}{2}$ (maximum, anticlock)

When $\theta = 90^\circ$, $\sigma_\theta = 0$ and $\tau_\theta = 0$

When $\theta = 135^\circ$, $\sigma_\theta = \sigma_x/2$ and $\tau_\theta = \frac{\sigma_x}{2}$ (maximum, clockwise)

Biaxial Stress Condition (or) two dimensional Stress Condition



Let an element of a body be acted upon by two tensile stresses acting on two perpendicular planes of the body as shown in fig. Let dx, dy and ds be the lengths of the sides AB, BC and AC respectively.

Considering unit thickness of the body and resolving
! the forces in the direction of σ_θ.

$$\sigma_\theta \cdot ds - \sigma_x \cdot dy \cdot \cos\theta - \sigma_y \cdot dx \cdot \sin\theta = 0$$

$$\sigma_\theta = \frac{\sigma_x \cdot dy \cos\theta}{ds} + \frac{\sigma_y \cdot dx \sin\theta}{ds}$$

$$\sigma_\theta = \frac{\sigma_x \, dy \cdot \cos\theta}{\frac{dy}{\cos\theta}} + \frac{\sigma_y \cdot dx \cdot \sin\theta}{\frac{dx}{\sin\theta}}$$

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$

The expression may be put in the following form

$$\sigma_\theta = \sigma_x \left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y \left(\frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta$$

Resolving the forces in the direction of τ_θ

$$\tau_\theta \cdot ds + \sigma_x \cdot dy \cdot \sin\theta - \sigma_y \cdot dx \cdot \cos\theta = 0$$

$$\tau_\theta = -\frac{\sigma_x \cdot dy \cdot \sin\theta}{ds} + \frac{\sigma_y \cdot dx \cdot \cos\theta}{ds}$$

$$= -\frac{\sigma_x \cdot dy \sin\theta}{dy/\cos\theta} + \frac{\sigma_y \cdot dx \cos\theta}{dx/\sin\theta}$$

$$= -\sigma_x \sin\theta \cdot \cos\theta + \sigma_y \cdot \sin\theta \cdot \cos\theta$$

$$= -(\sigma_x - \sigma_y)\sin\theta \cdot \cos\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

This expression indicates that it is counter-clockwise, If σ_x is more than σ_y.

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$

$$= \left[\left\{\frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta\right\}^2 + \left\{-\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta\right\}^2\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x + \sigma_y)^2 + \frac{1}{4}(\sigma_x - \sigma_y)^2\cos^2 2\theta + \frac{1}{2}(\sigma_x + \sigma_y)(\sigma_x - \sigma_y)\cos 2\theta + \frac{1}{4}(\sigma_x - \sigma_y)^2 \sin^2 2\theta\right]$$

$$= \left[\frac{1}{4}(\sigma_x + \sigma_y)^2 + \frac{1}{4}(\sigma_x - \sigma_y)^2 + \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \sqrt{\sigma_x^2 \cos^2\theta + \sigma_y^2 \sin^2\theta}$$

Angle of inclination,

$$\tan\phi = \frac{\tau_\theta}{\sigma_\theta} = \frac{-(\sigma_x - \sigma_y)\sin\theta \cdot \cos\theta}{\sigma_x \cos^2\theta + \sigma_y \sin^2\theta}$$

$$= \frac{\sigma_y - \sigma_x}{\sigma_x \cot\theta + \sigma_y \tan\theta}$$

The greast obliquity (or) inclination of resultant with the normal stress

$$\frac{d(\tan\phi)}{d\theta} = 0$$

$$\tan\theta = \sqrt{\frac{\sigma_x}{\sigma_y}}$$

$$\tan\phi_{max} = \frac{\sigma_y - \sigma_x}{\sigma_x\sqrt{\frac{\sigma_y}{\sigma_x}} + \sigma_y \cdot \sqrt{\frac{\sigma_x}{\sigma_y}}}$$

$$\tan(\phi_{max}) = \frac{\sigma_y - \sigma_x}{2\sqrt{\sigma_x \cdot \sigma_y}}$$

From the above result

→ The normal stress on the inclined plane varies between the values of σ_x and σ_y as the angle θ is increased from 0° to 90°. For equal values of the two axial stresses ($\sigma_x = \sigma_y$), σ_θ is always equal to σ_x (or) σ_y.

→ The shear stress is zero on planes with angles $0°$ and $90°$ i.e. on horizontal and vertical planes. It has maximum value numerically equal to one half the difference between given normal stresses which occurs on planes at $\pm 45°$ to the given planes

$$\tau_{max} = \pm \frac{1}{2}(\sigma_x - \sigma_y)$$

and the normal stress across the same plane

$$\sigma_{45°} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 90°$$

$$\sigma_{45°} = \frac{1}{2}(\sigma_x + \sigma_y)$$

→ Shear stress in a body subjected to two equal perpendicular stresses is zero.

→ If any of the given stresses is compressive, the stress can be replaced by a negative sign in the above derived expression. i.e. σ_x with $-\sigma_x$ & σ_y with $-\sigma_y$.

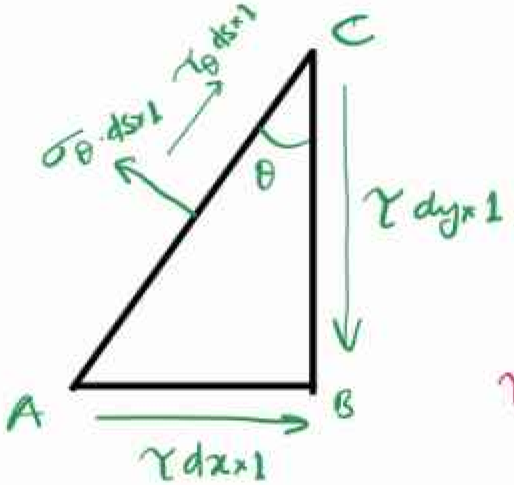
→ If σ_y is compressive, the maximum value of shear stress across a plane at $45°$

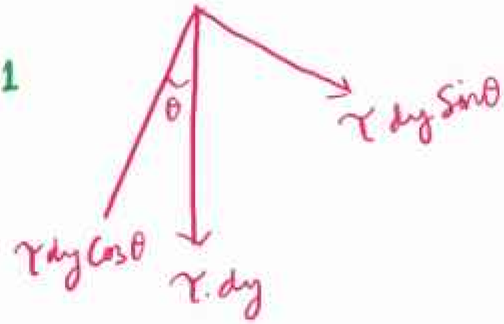
$$\tau_{max} = \frac{1}{2}\left[\sigma_x - (-\sigma_y)\right] = \frac{1}{2}(\sigma_x + \sigma_y)$$

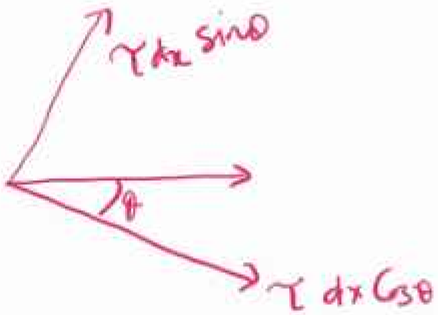
and σ_x is numerically equal to σ_y,

$$\tau_{max} = \sigma_x = \sigma_y.$$

Pure Shear Stress Condition







Let an element of a body be acted upon by shear stresses on its two perpendicular faces.

Let dx, dy & ds be the lengths of the sides AB, BC and AC respectively.

Considering unit thickness of the body and resolving forces in two directions

$$\sigma_\theta \cdot ds - \tau\, dx \cos\theta - \tau\, dy \sin\theta = 0$$

$$\sigma_\theta = \frac{\tau\, dx \cos\theta}{ds} + \frac{\tau\, dy \sin\theta}{ds}$$

$$\sigma_\theta = \frac{\tau\, dx \cos\theta}{\frac{dx}{\sin\theta}} + \frac{\tau\, dy \sin\theta}{\frac{dy}{\cos\theta}}$$

$$\sigma_\theta = \tau \sin\theta \cdot \cos\theta + \tau \sin\theta \cdot \cos\theta$$

$$\boxed{\sigma_\theta = \tau \sin 2\theta}$$

Resolving forces in the direction of τ_θ.

$$\tau_\theta \cdot ds - \tau\, dy \cdot \cos\theta + \tau\, dx \cdot \sin\theta = 0$$

$$\tau_\theta = \frac{\tau\, dy \cos\theta}{ds} - \frac{\tau\, dx \sin\theta}{ds}$$

$$= \frac{\tau\, dy \cos\theta}{dy / \cos\theta} - \frac{\tau\, dx \sin\theta}{\frac{dx}{\sin\theta}}$$

$$= \tau \cos^2\theta - \tau \sin^2\theta$$

$$= \tau \left[\left(\frac{1 + \cos 2\theta}{2} \right) - \left(\frac{1 - \cos 2\theta}{2} \right) \right] = \tau \cos 2\theta$$

$$\boxed{\tau_\theta = \tau \cos 2\theta}$$

From this derivation it is shown that the pure shear will be up the plane for $\theta < 45^\circ$ and down the plane for $\theta > 45^\circ$

The resultant stress on the plane AC,

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \tau\sqrt{(\sin 2\theta)^2 + (\cos 2\theta)^2}$$

$$\boxed{\sigma_r = \tau}$$

Inclination with the direction of shear stress plane,

$$\tan\varphi = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

$$\boxed{\varphi = 2\theta}$$

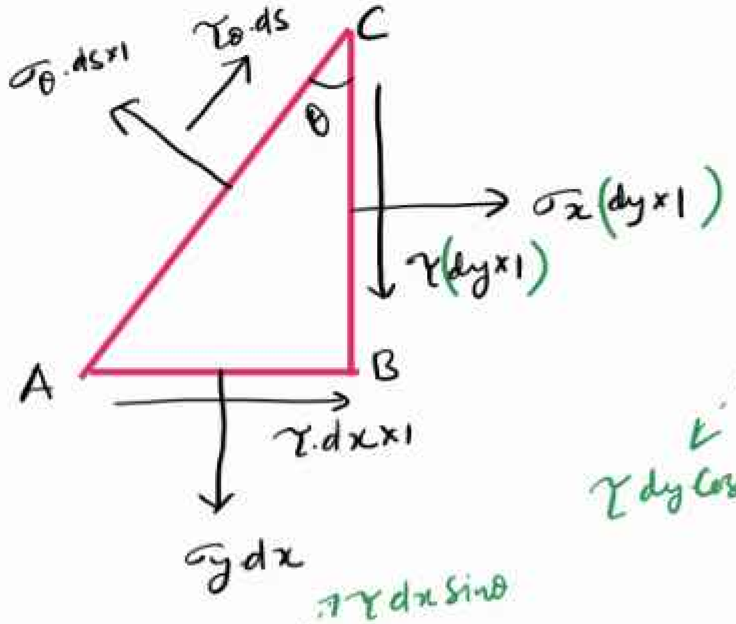
Conclusions from above eq^ns

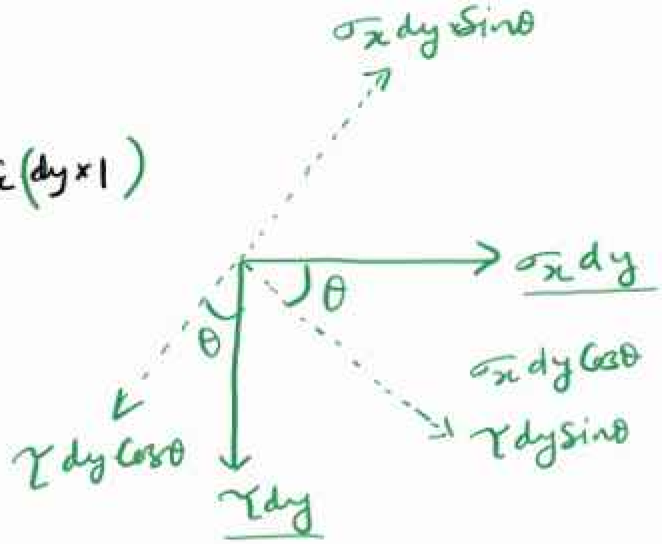
1. The normal stress is positive (tensile) when $\theta \simeq 0^\circ$ to 90° and (-ve) compressive between 90° to 180°. Maximum value at 45° (τ) and 135° $(-\tau)$

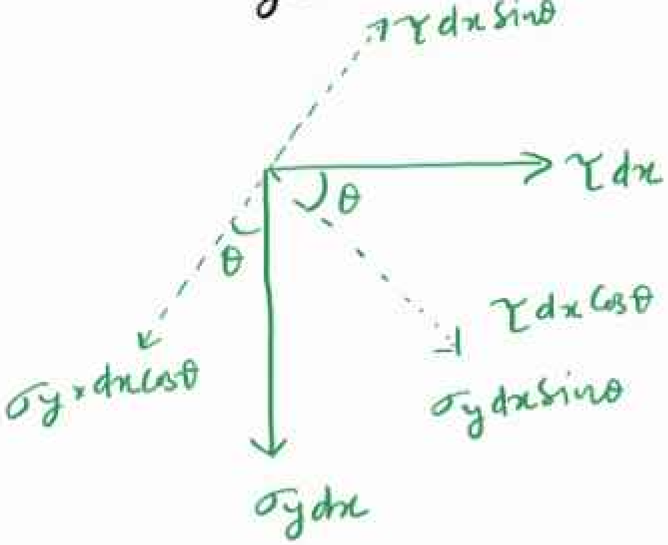
2. The shear stress is (+ve) for $\theta < 45^\circ$ and negative (anticlockwise) for $\theta > 45^\circ$ and $< 135^\circ$ and again positive $\theta > 135^\circ$ and $< 180^\circ$

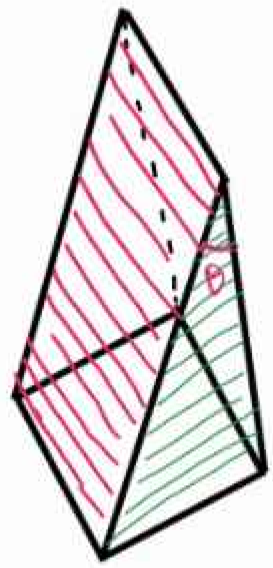
3. Shear stress is zero at 45° and 135° where the normal stress is maximum.

Biaxial and Shear Stress Condition









Let an element of a body be acted by two tensile stresses along with shear stresses acting on two perpendicular planes of the body. Let dx, dy & ds be the lengths of the sides AB, BC & AC.

Considering thickness of the body and resolving the forces in the direction of σ_θ.

$$\sigma_\theta \cdot ds = \sigma_x \cdot dy \cdot \cos\theta + \sigma_y \cdot dx \cdot \sin\theta + \tau dy \sin\theta + \tau dx \cos\theta$$

$$\sigma_\theta = \frac{\sigma_x dy \cos\theta}{ds} + \frac{\sigma_y \cdot dx \sin\theta}{ds} + \frac{\tau dy \sin\theta}{ds} + \frac{\tau dx \cos\theta}{ds}$$

$$\sigma_\theta = \frac{\sigma_x dy \cos\theta}{dy/\cos\theta} + \frac{\sigma_y . dx \sin\theta}{dx/\sin\theta} + \frac{\tau dy \sin\theta}{dy/\cos\theta} + \frac{\tau dx \cos\theta}{dx/\sin\theta}$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin\theta \cdot \cos\theta + \tau \sin\theta \cdot \cos\theta$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta$$

$$= \sigma_x \left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y \left(\frac{1-\cos 2\theta}{2}\right) + \tau \sin 2\theta$$

$$\boxed{\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \sin 2\theta}$$

Resolving the forces in the direction of τ

$$\tau_\theta ds + \sigma_x dy \sin\theta - \sigma_y dx \cos\theta - \tau dy \cos\theta + \tau dx \sin\theta = 0$$

$$\tau_\theta = -\frac{\sigma_x dy \sin\theta}{ds} + \frac{\sigma_y dx \cos\theta}{ds} + \frac{\tau dy \cos\theta}{ds} - \frac{\tau dx \sin\theta}{ds}$$

$$= -\frac{\sigma_x dy \sin\theta}{dy/\cos\theta} + \frac{\sigma_y dx \cos\theta}{dx/\sin\theta} + \frac{\tau dy \cos\theta}{dy/\cos\theta} - \frac{\tau dx \sin\theta}{dx/\sin\theta}$$

$$= -\sigma_x \sin\theta \cdot \cos\theta + \sigma_y \sin\theta \cdot \cos\theta + \tau \cos^2\theta - \tau \sin^2\theta$$

$$= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\right]$$

$$\boxed{\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta}$$

To determine the planes having maximum and minimum values of direct stress, differentiate the above eqn w.r.t. θ and equate to zero.

$$\frac{d\sigma_\theta}{d\theta} = 0 \Rightarrow -\frac{1}{2}(\sigma_x - \sigma_y) \times 2\sin 2\theta + 2\tau\cos 2\theta = 0$$

$$\frac{1}{2}(\sigma_x - \sigma_y)\, 2\sin 2\theta = 2\tau\cos 2\theta$$

$$\boxed{\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}}$$

This equation provides two values of 2θ differing by $180°$ (or) θ by $90°$, the planes along which the direct stresses have the maximum and minimum values.

Wherever maximum (or) minimum value of direct stresses are there, at the same place shear stresses are zero.

SUM OF DIRECT STRESSES ON TWO MUTUALLY PERPENDICULAR PLANES

Direct stress on an inclined plane at angle θ is given by

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau\sin 2\theta \quad \text{--- (1)}$$

Direct stress on inclined plane at angle $(\theta + 90°)$

$$\sigma_{(\theta+90°)} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2(\theta + 90°) + \tau\sin 2(\theta + 90°)$$

$$= \frac{1}{2}(\sigma_x+\sigma_y) - \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta - \tau \sin 2\theta \quad \text{—— (2)}$$

Adding (1) & (2)

$$\boxed{\sigma_\theta + \sigma_{(\theta+90^\circ)} = \sigma_x + \sigma_y}$$

PRINCIPAL STRESSES

In general, a body may be acted upon by direct stresses and shear stresses. However, it will be seen that even in such complex system of loadings, there exist three mutually perpendicular planes, on each of which the resultant stress is wholly normal.

These are known as principal planes and the normal stress across these planes, as principal stresses. The corresponding planes are known as major and minor principal planes.

In two dimensional problem, the third principal stress is taken to be zero.

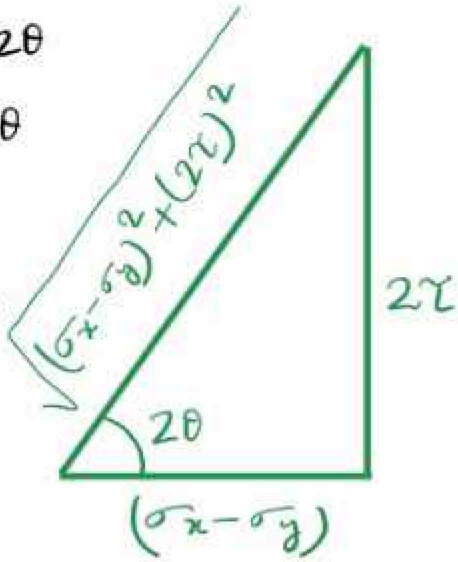
As consider, shear stress is zero in principal planes

$$\tau_\theta = -\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta + \tau\cos 2\theta = 0$$

$$\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta = \tau\cos 2\theta$$

$$\boxed{\tan 2\theta = \frac{2\tau}{(\sigma_x-\sigma_y)}}$$

Which provides two values of 2θ differing by 180° (or) two values of θ differing by 90°. Thus, the two principal planes are perpendicular to each other



$$\sin 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}}$$

$$\cos 2\theta = \pm \frac{\sigma_x-\sigma_y}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}}$$

Right hand sides of both the above equations should have the same signs, positive (or) negative. Substituting these values of $\sin 2\theta$ and $\cos 2\theta$ in eqn (1)

there are two values of the direct stresses, i.e. of principal stresses corresponding to two values of 2θ are obtained.

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x+\sigma_y) + \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta + \tau \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x+\sigma_y) \pm \frac{1}{2}\frac{(\sigma_x-\sigma_y)^2}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}} \pm \tau \cdot \frac{2\tau^2}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}}$$

$$= \frac{1}{2}(\sigma_x+\sigma_y) \pm \frac{1}{2}\frac{(\sigma_x-\sigma_y)^2+4\tau^2}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}}$$

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

σ_1 = Major principal stress

σ_2 = Minor principal stress

$$\boxed{\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$\boxed{\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

MAXIMUM PRINCIPAL SHEAR STRESSES

In any complex system of loading the maximum and minimum normal stresses are the principal stresses and the shear stresses is zero in their plane. To find maximum value of shear stress and its plane in such a system,

Consider the equation for shear stress

$$\boxed{\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta}$$

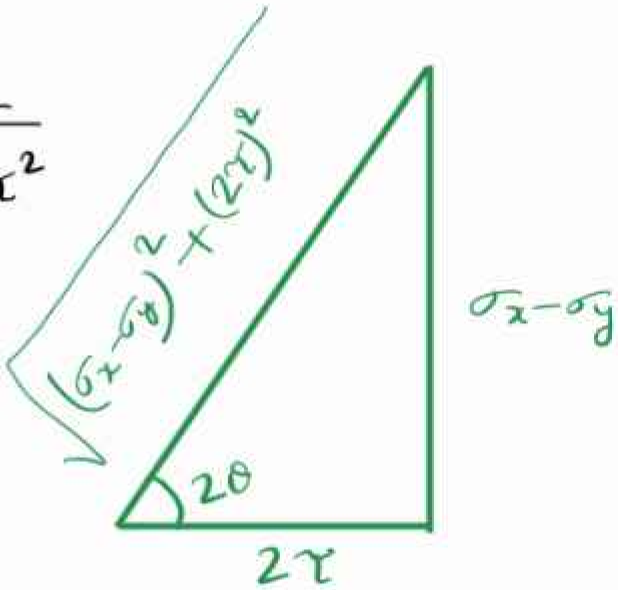
For maximum value of τ_θ, differentiate it w.r.t. θ and equate to zero

$$\frac{d\tau_\theta}{d\theta} = -\frac{1}{2} \times 2(\sigma_x - \sigma_y)\cos 2\theta - 2\tau \sin 2\theta = 0$$

$$\tan 2\theta = -\left(\frac{\sigma_x - \sigma_y}{2\tau}\right)$$

$$\sin 2\theta = \mp \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$\cos 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$



$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta$$

$$= -\left(\mp \frac{1}{2}(\sigma_x - \sigma_y)\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}\right) \pm \tau \frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \pm \frac{1}{2}\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

This provides maximum and minimum values of shear stress, both numerically equal. In fact, the negative or minimum value indicates that it is at right angles to the positive value. and two are the complimentary shear stresses.

Thus, the magnitude of maximum (or) principal shear stress is given by

$$\boxed{\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}}$$

As it is given,

Maximum principal stress, $\sigma_1 = \frac{1}{2}(\sigma_x+\sigma_y)+\frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}$ — (1)

Minimum principal stress, $\sigma_2 = \frac{1}{2}(\sigma_x+\sigma_y)-\frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}$ — (2)

from (1) & (2)

$$\sigma_1-\sigma_2 = 2\times\underbrace{\left\{\frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}\right\}}_{\tau_{max}}$$

$$\sigma_1-\sigma_2 = 2(\tau_{max})$$

$$\boxed{\tau_{max} = \frac{\sigma_1-\sigma_2}{2} = \frac{1}{2}(\sigma_1-\sigma_2)}$$

In general, $\tau_{max} = \frac{1}{2}(\sigma_1-\sigma_2) = \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}$

As derived, principal planes are given by

$$\tan 2\theta_p = \frac{2\tau}{\sigma_x-\sigma_y}$$

planes of maximum shear stress, $\tan 2\theta_s = -\frac{\sigma_x-\sigma_y}{2\tau}$

Multiplying these two,

$$\tan 2\theta_p \times \tan 2\theta_s = -1$$

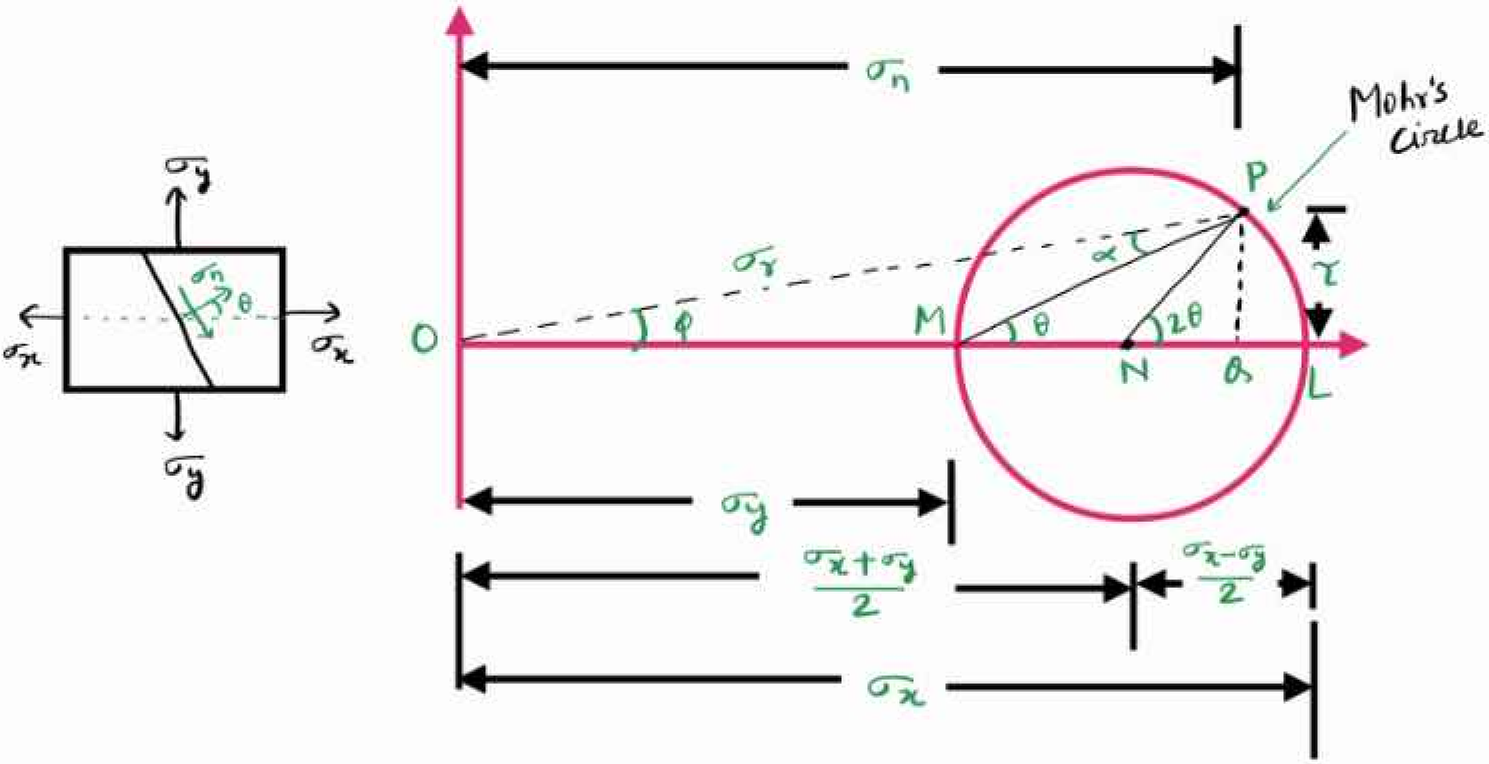
it means, $2\theta_s = 2\theta_p + 90^\circ$ (or) $\theta_s = \theta_p + 45^\circ$

This indicates that the planes of maximum shear stress lies at 45° to the planes of principal axes.

MOHR'S STRESS CIRCLE

Otto Mohr devised a graphical method for finding out the normal and shear stresses on any interface of an element when it is subjected to two perpendicular stresses.

(1) Mohr's circle construction for 'like stresses'



① $OL = \sigma_x$

$OM = \sigma_y$

② Bisect LM at M

(3) Considering N as centre and NL (or) NM radius, draw a circle.

(4) At the centre N draw a line NP at angle 2θ

(5) From P, drop a perpendicular line PQ on the axis OX.

PQ will represent τ as OQ σ_n.

$$NP = NL = \frac{\sigma_x - \sigma_y}{2}$$

$$PQ = NP \sin 2\theta = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta = \tau$$

Similarly, $OQ = ON + NQ = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta = \sigma_n$

$$\boxed{\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta}$$

from stress circle, τ is maximum when

$$2\theta = 90^\circ \text{ (or) } \theta = 45^\circ$$

and $\boxed{\tau_{max} = \frac{\sigma_x - \sigma_y}{2}}$

Sign conventions:

(i) To mark τ in stress system, we will take the clockwise shear as positive and anticlockwise shear as negative.

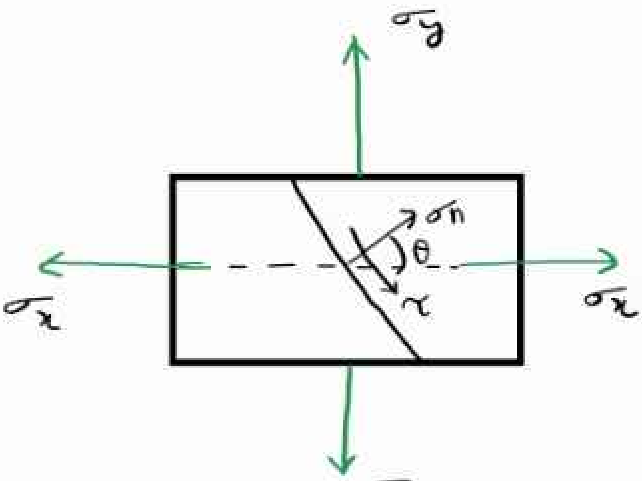
(ii) Positive value of τ will be above the axis and negative value below the axis.

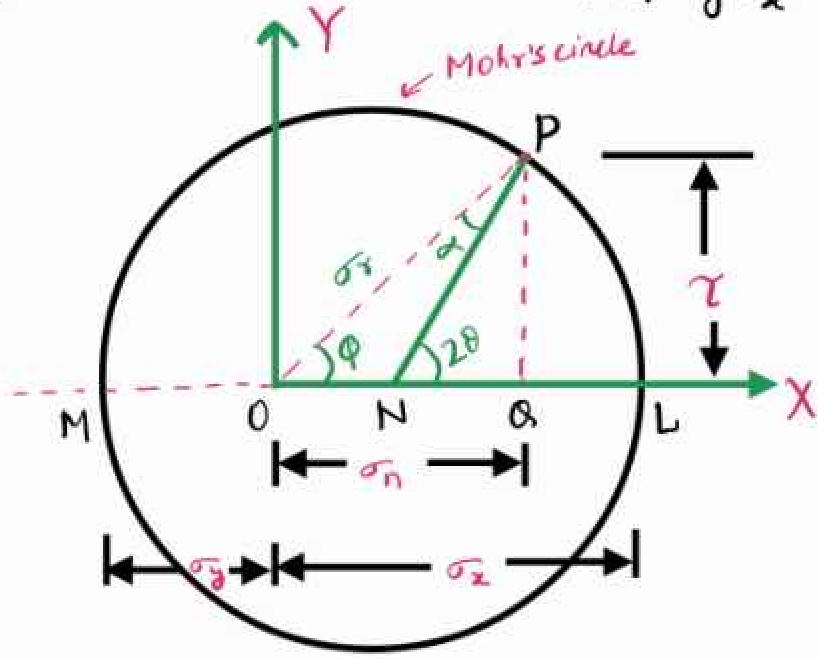
(iii) If θ is in the anticlockwise direction, the radius vector will be the above the axis and θ will be rekoned positive. If θ is in clockwise direction, it will be negative and the radius vector will be below axis.

(IV) Tensile stress will be rekoned positive and will be plotted to the right of the origin O. Compressive stress will be rekoned negative and will be plotted to the left of the origin O.

(2) Mohr's circle construction for 'unlike stresses'.

In case σ_x & σ_y are not like, the same procedure will be followed except that σ_x and σ_y will be measured to the opposite sides of the origin. It may be noted that the direction of σ_n will depend upon its position w.r.t. the point O. If it is to the right of O, the direction of σ_n will be the same as that of σ_x.

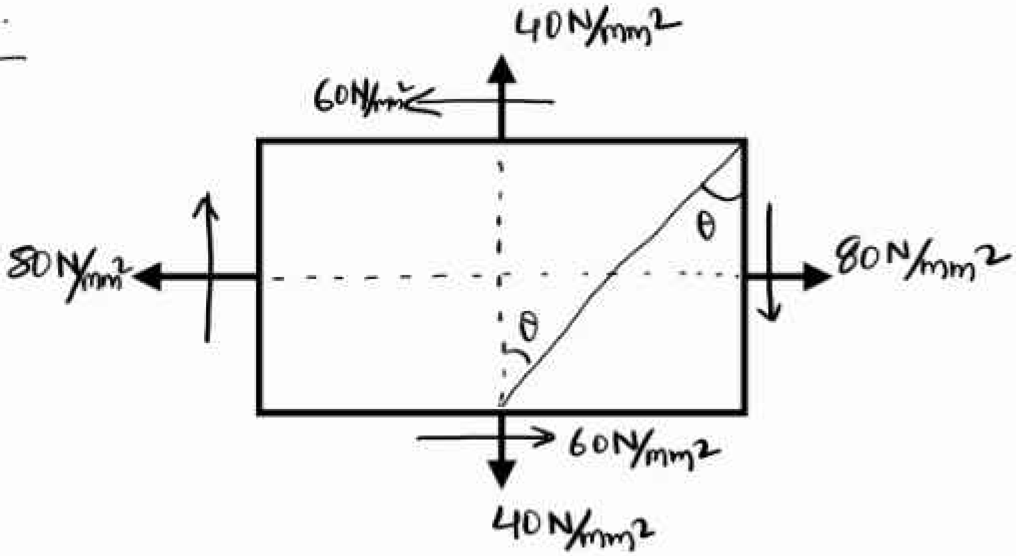




Problems:

At a point within a body subjected to two mutually perpendicular directions, the stresses are 80N/mm² tensile and 40N/mm² tensile. Each of above stresses is accompained by a shear stress of 60N/mm². Determine the normal stress, shear Stress and resultant stress on an oblique plane inclined at an angle of 45° with the axis of minor tensile stress.

Soln.



$\sigma_x = 80 N/mm^2$

$\sigma_y = 40 N/mm^2$

$\tau = 60 N/mm^2$

Angle of oblique plane, with the axis of minor tensile stress,

$$\theta = 45^\circ$$

Normal stress (σ_n)

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \sin 2\theta$$

$$= \frac{1}{2}(80+40) + \frac{1}{2}(80-40)\cos 90^\circ + 60 \times \sin 90^\circ$$

$$= \frac{120}{2} + \frac{40}{2} \times 0 + 60 \times 1$$

$$\boxed{\sigma_n = 60 + 60 = 120 \ N/mm^2}$$

Shear Stress

$$\tau_\theta = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau \cos 2\theta$$

$$= -\left(\frac{80-40}{2}\right)\sin(90^\circ) + 60 \times \cos 90^\circ$$

$$= -\left(\frac{40}{2}\right) \times 1 + 0$$

$$\tau_\theta = -20 \ N/mm^2 \ \text{(Anticlock direction)}$$

Resultant Stress

$$\sigma_r = \sqrt{\sigma_n^2 + \tau_\theta^2}$$

$$= \sqrt{120^2 + (20)^2} = \sqrt{120 \times 120 + 400}$$

$$= \sqrt{14400 + 400} = \sqrt{14800}$$

$$\boxed{\sigma_r = 121.655 \ N/mm^2}$$

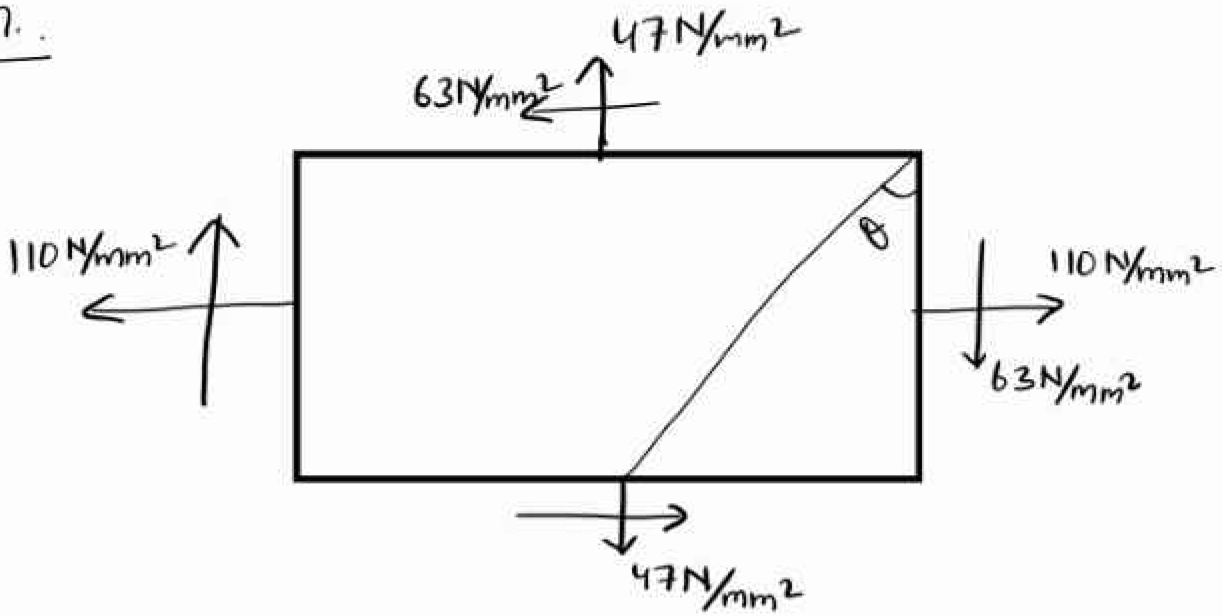
Problem

A rectangular block of material is subjected to a tensile stress of 110 N/mm² on one plane and a tensile stress of 47 N/mm² on the plane at right angles to the former. Each of the above stresses is accompanied by a shear stress of 63 N/mm² and that associated with the former tensile stress tend to rotate the block anticlockwise.

Find (i) The direction and magnitude of each of the principal stress and

(ii) Magnitude of the greatest shear stress.

Soln.



$\sigma_x = 110\ N/mm^2$

$\sigma_y = 47\ N/mm^2$

$\tau = 63\ N/mm^2$

Major principal stress $(\sigma_1) = \dfrac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$

$$= \frac{110+47}{2} + \sqrt{\left(\frac{110-47}{2}\right)^2 + 63^2}$$

$$= \frac{157}{2} + \sqrt{992.25 + 3969}$$

$$\sigma_1 = 78.5 + 70.44 = 148.936 \text{ N/mm}^2$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

$$= \frac{110+47}{2} - \sqrt{\left(\frac{110-47}{2}\right)^2 + 63^2}$$

$$\sigma_2 = 78.5 - 70.44 = 8.06 \text{ N/mm}^2$$

Direction of principal stresses

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}$$

$$= \frac{2 \times 63}{110-47}$$

$$\tan 2\theta = 2$$

$$2\theta = 63.43$$

$$\theta = 31^\circ 43' \quad (\text{or}) \quad 121^\circ 43'$$

Magnitude of greatest shear stress

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(110-47)^2 + 4 \times 63^2}$$

$$= \frac{1}{2}\sqrt{63^2 + 4\times 63^2} = \frac{1}{2}\sqrt{63^2 + 2^2\times 63^2}$$

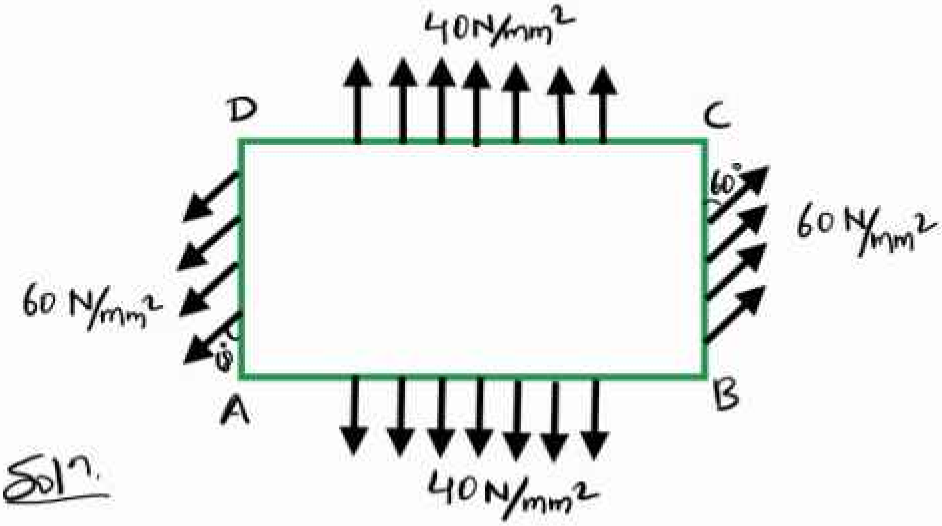
$$= \frac{1}{2}\times 63\sqrt{1+4}$$

$$= \frac{63}{2}\sqrt{5}$$

$$\boxed{\tau_{max} = 70.436\ N/mm^2}$$

A point in a strained material is subjected to to the stresses as shown in figure.

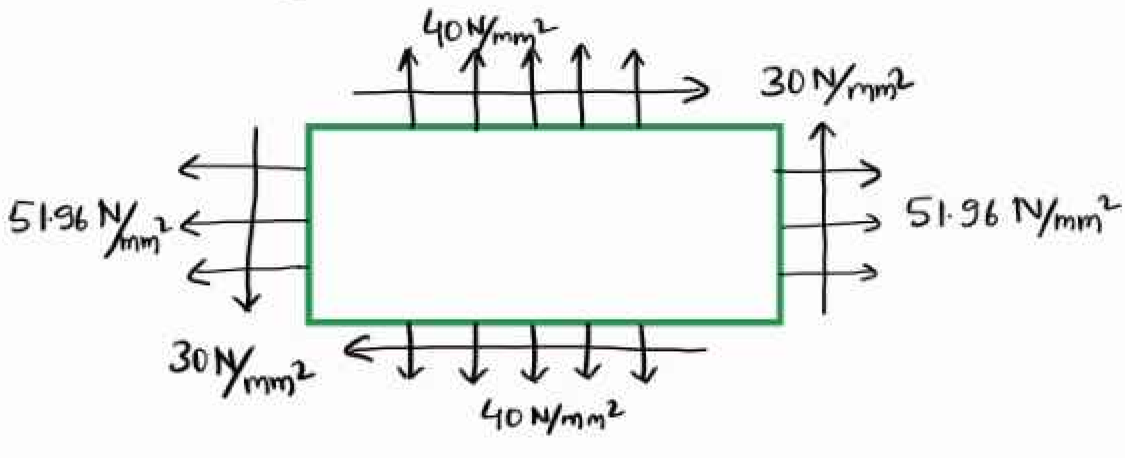
Locate the principal planes, and evaluate the principal stresses.



Soln.

Normal Stress on face BC & AD $= 60\times \sin 60° = 51.96\ N/mm^2$

Stress along BC & AD $= 60\times \cos 60° = 60\times \frac{1}{2} = 30\ N/mm^2$



Major tensile stress, $\sigma_x = 51.96\ N/mm^2$

Minor tensile stress, $\sigma_y = 40\ N/mm^2$

Shear stress, $\tau = 30\ N/mm^2$

Location of principal planes

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}$$

$$= \frac{2 \times 30}{51.96 - 40} = 5.0167$$

$$2\theta = 78.7268$$

$$\theta = 39.36^\circ = 39^\circ 21' \text{ (or) } 129^\circ 21'$$

Principal Stress

∴ Major principal Stress

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

$$= \frac{51.96 + 40}{2} + \sqrt{\left(\frac{51.56 - 40}{2}\right)^2 + 30^2}$$

$$= 45.98 + 30.6$$

$$\boxed{\sigma_1 = 76.58\ N/mm^2}$$

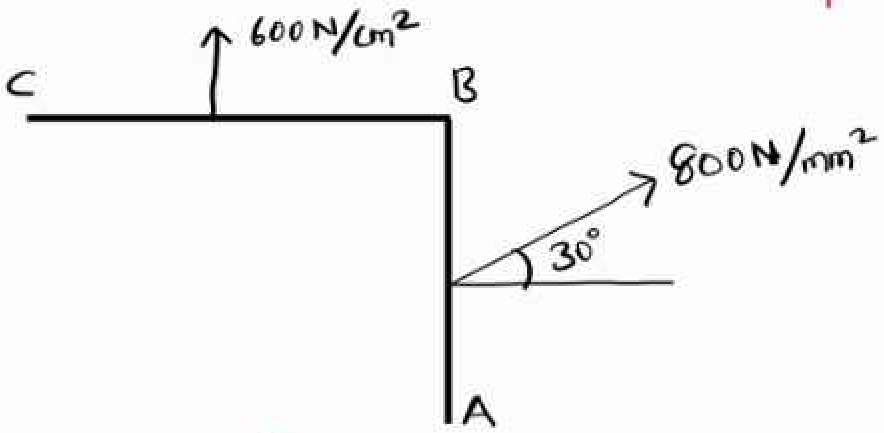
Minor principal stress

$$\sigma_2 = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau^2}$$

$$= \frac{51.96+40}{2} - \sqrt{\left(\frac{51.96-40}{2}\right)^2 + 30^2}$$

$$\boxed{\sigma_2 = 45.98 - 30.6 = 15.38 \text{ N/mm}^2}$$

The intensity of resultant stress on a plane AB at a point in a material under stress is 800 N/cm² and it is inclined at 30° to the normal to that plane. The normal component of stress on another plane BC at right angles to plane AB is 600 N/cm².



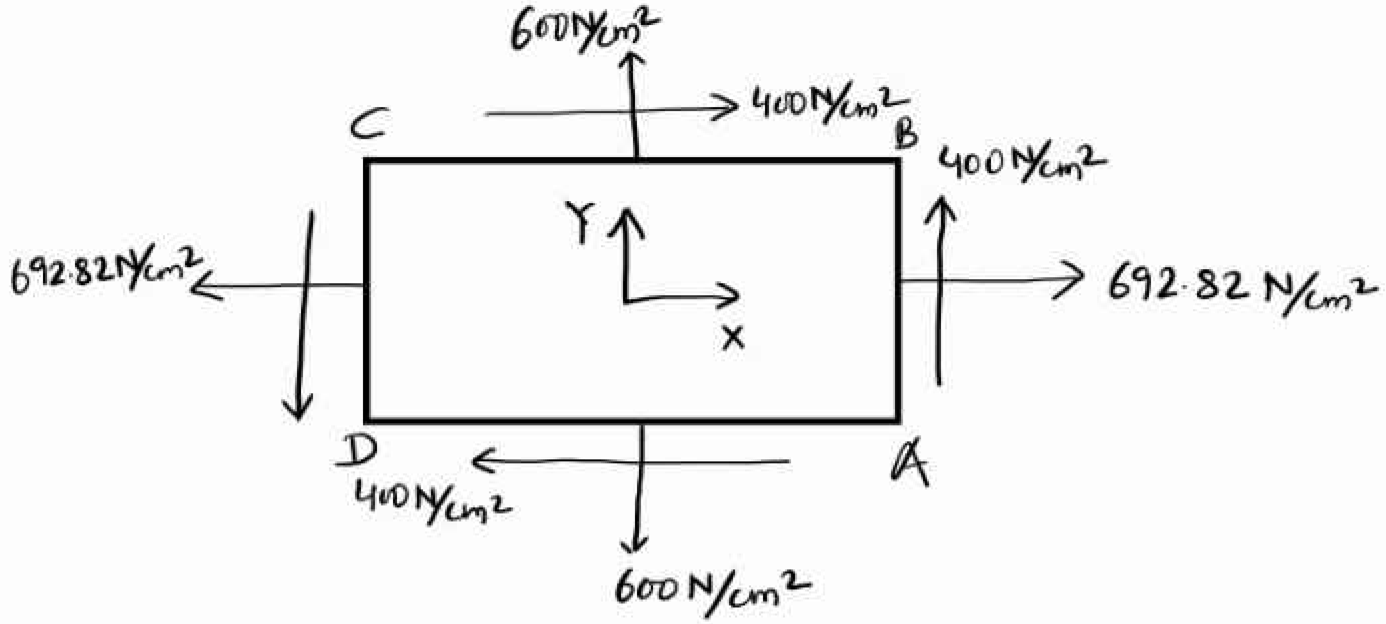
Determine the following:-

(i) Resultant stress on plane BC

(ii) The principal stress and their directions,

(iii) The maximum shear stresses and their planes.

Sol^n.



The normal stress on plane AB = $800 \times \cos 30^\circ = 692.82$ N/cm²

The tangential stress on plane AB = $800 \times \sin 30^\circ = 400$ N/cm²

$$\sigma_x = 692.82 \text{ N/cm}^2$$

$$\sigma_y = 600 \text{ N/cm}^2$$

$$\tau = 400 \text{ N/cm}^2$$

(1) Resultant stress on plane BC $= \sqrt{\sigma_y^2 + \tau^2}$

$$= \sqrt{600^2 + 400^2}$$

$$= 721 \text{ N/cm}^2$$

The resultant will be inclined at an angle θ with the horizontal

$$\tan\theta = \frac{\sigma_y}{\tau} = \frac{600}{400} = 1.5$$

$$\theta = \tan^{-1}(1.5) = 56.31 = 56^\circ 18'$$

② Principal stresses and their directions

Major principal stress, $\sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau^2}$

$$= \frac{692.82+600}{2} + \sqrt{\left(\frac{692.82-600}{2}\right)^2 + 400^2}$$

$$\sigma_1 = 646.41 + 402.68 = 1049.09 \text{ N/cm}^2 \text{ (tensile)}$$

Minor principal stress,

$$\sigma_2 = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau^2}$$

$$= \frac{692.82+600}{2} - \sqrt{\left(\frac{692.82-600}{2}\right)^2 + 400^2}$$

$$= 646.41 - 402.68$$

$$\sigma_2 = 243.73 \text{ N/cm}^2$$

The directions of principal stresses

$$\tan 2\theta = \frac{2\tau}{\sigma_x-\sigma_y} = \frac{2\times 400}{(692.82-600)^2} = \frac{800}{92.82} = 8.618$$

$$2\theta = \tan^{-1}(8.618) = 83.38^\circ$$

$$\theta = 41^\circ 41' \text{ (or) } 131^\circ 41'$$

(iii) The maximum shear stress and their planes.

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(692.82 - 600)^2 + 4 \times 400^2}$$

$$= \frac{1}{2}\sqrt{8615.55 + 640000}$$

$$\boxed{\tau_{max} = 402.68 \text{ N/cm}^2}$$

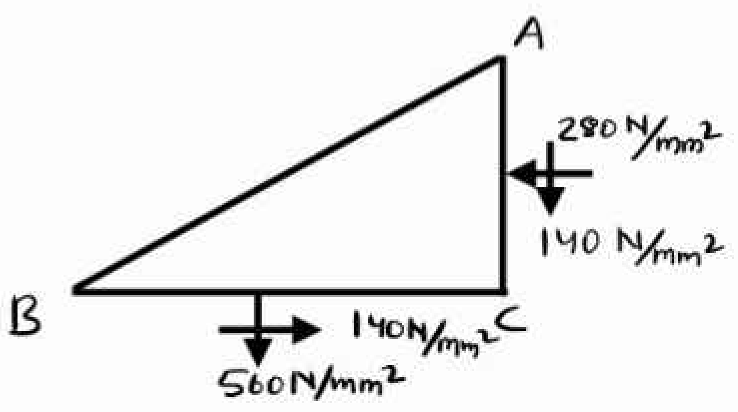
Practice problem

At a point in a strained material, on plane BC there are normal and shear stresses of 560 N/mm² and 140 N/mm² respectively. One plane AC, perpendicular to plane BC, there are normal and shear stresses of 280 N/mm² and 140 N/mm² respectively as shown in figure.

Determine.

(i) Principal stresses and location of the planes on which they act.

(ii) Maximum shear stress and the plane on which it act.



NUMERICAL PROBLEM BASED ON MOHR's CIRCLE

The tensile stresses at a point mutually two perpendicular planes are 120 N/mm^2 and 60 N/mm^2. Determine the normal, tangential and resultant stresses on a plane inclined at 30° to the axis of minor stresses. Solve this problem by using Mohr's circle method.

Sol^n.

$\sigma_1 = 120\ N/mm^2$

$\sigma_2 = 60\ N/mm^2$

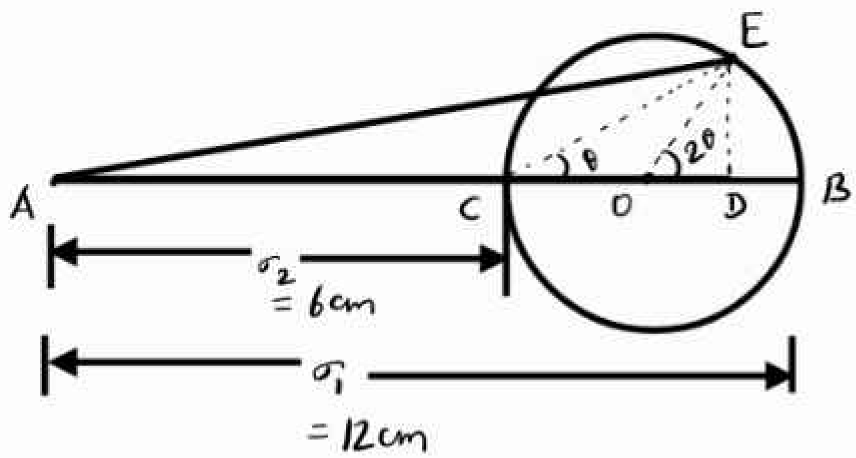
$\theta = 30^\circ$

Assume a scale, $1\ cm = 10\ N/mm^2$

So, $\sigma_1 = 12\ cm$

$\sigma_2 = 6\ cm$

Now draw a Mohr's circle



diameter of the circle $= \sigma_1 - \sigma_2 = 12 - 6 = 6\ cm$

O is the centre of the circle.

Through O, draw a line OE making an angle 2θ

i.e $(2 \times 30^\circ = 60^\circ)$ with OB

From E, draw ED ⊥lar to CB

Join AE, measure length AD, ED and AE

$$\text{Radius of circle } OE = \frac{6}{2} = 3\text{ cm}$$

$$\cos 2\theta = \frac{OD}{OE}$$

$$OD = OE \cos 2\theta = 3 \times \cos 60^\circ = 3 \times \frac{1}{2} = 1.5\text{ cm}$$

$$\text{So, } AD = 1.5 + 3 + 6 = 10.5\text{ cm}$$

$$\sin 2\theta = \frac{ED}{OE}$$

$$\text{length of } ED = OE \sin 2\theta$$

$$= 3 \times \sin 60^\circ = 3 \times \frac{\sqrt{3}}{2} = 2.596 \simeq 2.6\text{ cm}$$

$$\text{Length of } AE = \sqrt{10.5^2 + 2.6^2} = \sqrt{}$$

$$= 10.82\text{ cm}$$

Normal stress = length of AD × Scale

$$= 10.5 \times 10 = 105\text{ N/mm}^2$$

Shear stress = τ = length of ED × 10

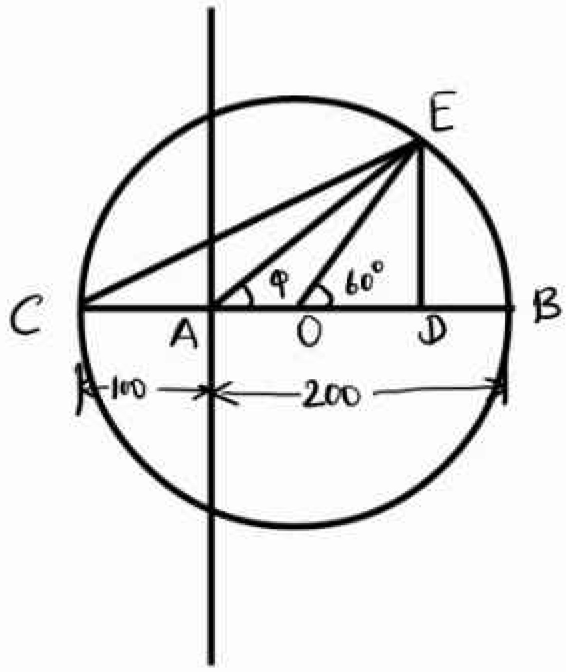
$$= 2.6 \times 10 = 26\text{ N/mm}^2$$

Resultant stress = Length of AE × Scale

$$= 10.82 \times 10 = 10.82\text{ N/mm}^2$$

The stresses at a point in a bar are 200 N/mm² (tensile) and 100 N/mm² (Compressive). Determine the resultant stress in magnitude and direction on a plane inclined at 60° to the axis of major stress. Also determine the maximum intensity of shear stress in the material at a point. Solve the problem using Mohr's circle.

Soln.



Given that

$\sigma_1 = 200 \ N/mm^2$

$\sigma_2 = -100 \ N/mm^2$ (Compressive)

$\theta = 30°$

Assum the scale, 1 cm = 20 N/mm²

$\sigma_1 = \frac{200}{20} = 10 \ cm$

$\sigma_2 = -\frac{100}{20} = -5 \ cm$

Take any point A and draw a horizontal line through A on both sides of A. Take $AB = \sigma_1 = 10cm$ towards right of A and $AC = -5cm$ towards left of A. Bisect BC at O. With O as centre and radius equal to CO (or) OB, draw a circle. Through O draw a line OE making an angle 2θ. i.e. $2\theta = 2 \times 30 = 60°$

Here, A join with E & B i.e. AE = AB, AE represent the resultant stress and φ represents the obliquity.

Here, OE = OB

$$OB = \frac{CA + AB}{2} = \frac{5+10}{2} = 7.5cm$$

$$\sin 60° = \frac{ED}{OE} =$$

$$ED = OE \sin 60° = 7.5 \times \sin 60°$$
$$= 7.5 \times \frac{\sqrt{3}}{2}$$

$$ED = 6.495 \simeq 6.5cm$$

$$\text{length of } AD = \text{length of } (OD + OA) \qquad \left\{ \cos\theta = \frac{OD}{OE} \right\}$$
$$= OE\cos\theta + \{OB - CA\}$$
$$= 7.5 \times \cos 60° + (7.5 - 5)$$
$$AD = 7.5 \times \frac{1}{2} + 2.5 = 3.75 + 2.5 = 6.25cm$$

$$\tan\varphi = \frac{6.5}{6.25} = 46°$$

Resultant stress = length of AE x Scale

$$\text{length of } AE = \frac{AD}{\cos\varphi} = \frac{6.25}{\cos 46^\circ} = 8.997 \text{ cm}$$

$$= 9 \text{ cm}$$

$$\text{Resultant stress} = 8.997 \times 20 = 179.94 \text{ N/mm}^2$$

$$\simeq 180 \text{ N/mm}^2$$

Normal stress = length of AD x Scale

$$= 6.25 \times 20 = 125 \text{ N/mm}^2$$

Shear stress = length of ED x 20

$$= 6.5 \times 20 = 130 \text{ N/mm}^2$$

The max. shear stress is equal to radius of mohr's circle.

$$\boxed{\text{So, } \tau_{max} = \frac{\sigma_1 + \sigma_2}{2} = \frac{200+100}{2} = 150 \text{ N/mm}^2}$$